Strict Lyapunov Function for Non-Smooth PI Controller

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System

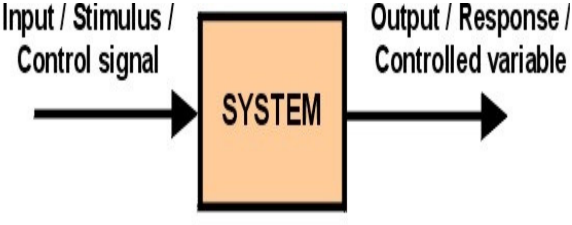


Figure: Any General System

Assumptions: Minimum Phase System

Minimum Phase System

A System is said to be Minimum-Phase if the system and its inverse are causal and stable. A system with rational transfer function is minimum-phase if all its zeros are also on the left half plane in addition to the poles.

(a) Given a specific output one can get the control explicitly on the r^{th} derivative (also known as relative degree).

(b) Feedback Linearization(non-linear version of pole zero cancellation) can't be used if the zero dynamics are unstable i.e. for non-minimum phase systems.

(c) System can be represented in chain of integrator form.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &= u. \end{aligned} \tag{1}$$

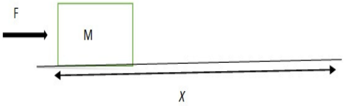
Relative Degree

Minimum Phase System

$$\begin{aligned} y &= x \\ \dot{y} &= \dot{x} \\ \ddot{y} &= \ddot{x} = \frac{f}{m}. \end{aligned} \tag{2}$$

1. In the below system friction dynamics is ignored.
2. Relative degree of the system *w.r.t x i.e. displacement* is 2.

The typical situations of the above problem is represented as the following:



Minimum and Non-Minimum Phase system example:

Consider the following systems

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3^2 + u \\ \dot{x}_3 &= -x_3. \end{aligned} \tag{3}$$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3^2 + u \\ \dot{x}_3 &= x_3. \end{aligned} \tag{4}$$

If the control input u in the systems 3 and 4 is taken as $-x_3^2 - k_1x_1 - k_2x_2$, the state x_1 and x_2 would go to zero. However, the zero dynamics of system 4 won't be stable. Hence System 4 is not a Minimum Phase System.

PID Controller and it's popularity

- Model-free and require minimal background.
- Capable of shaping systems both transient(PD part) and asymptotic performance(Integral Part).
- **Proportional part accounts for the present while the Integral and Derivative part accounts for the past and future respectively.**

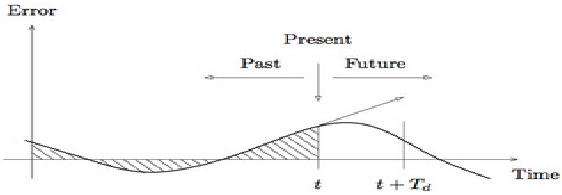


Figure: Action of a PID controller. At time t, the proportional term depends on the instantaneous value of the error. The integral portion of the feedback is based on the integral of the error up to time t (shaded portion). The derivative term provides an estimate of the growth or decay of the error over time by looking at the rate of change of the error.

PI Controller in industry

PI Controllers are popular in industry since derivative action is sensitive to noise.

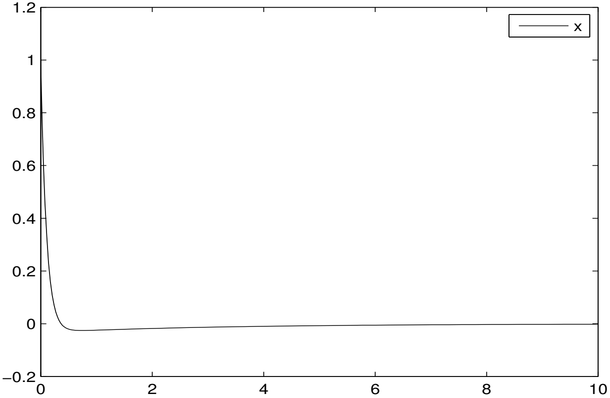
The great popularity of PID Controller goes hand in hand with their widespread misuse.

People often use PI controller in their system blindly and end up getting the undesired result.

?????????????????????Reason??????????????????????

One reason is the presence of time varying disturbance.

Response of PI Controller in presence of constant disturbance



PI Controller in presence of constant disturbance

Let the disturbance d in the following system be constant

$$\begin{aligned} \dot{x} &= u + d \\ u &= -k_p x - k_I \int_0^{\tau} x(\tau)\, d\tau \end{aligned} \tag{5}$$

Taking $z = -k_I \int_0^{\tau} x(\tau)\, d\tau + d$ the system 5 gets transformed to:

$$\begin{aligned} \dot{x} &= -k_p x + z \\ \dot{z} &= -k_I x \end{aligned} \tag{6}$$

In changed co-ordinate, the system becomes

$$\begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} -k_p & 1 \\ -k_I & 0 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \tag{7}$$

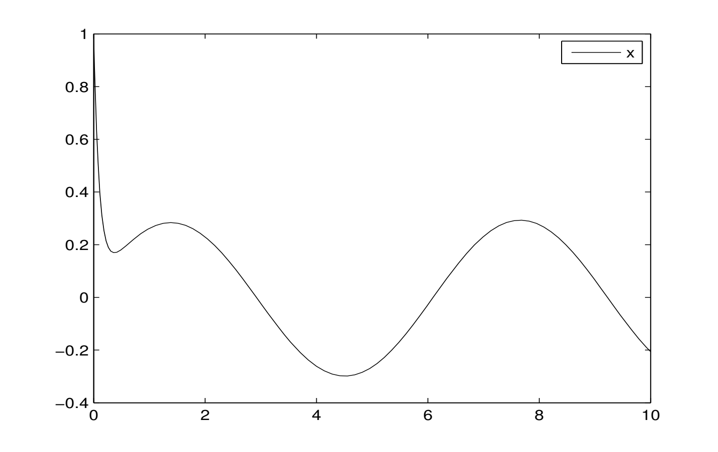
PI Controller in presence of constant disturbance

A transform T is applied on the state vector $\mathbf{X}$ such that $\mathbf{TX} = \mathbf{P}$. Thus, $\dot{\mathbf{P}} = \mathbf{TAT^{-1}P} = \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$, where λ_1 and λ_2 are the eigen values of the state matrix. Thus,

$$\begin{bmatrix} \dot{P}_1 \\ \dot{P}_2 \end{bmatrix} = \begin{bmatrix} -\lambda_1 P_1 \\ -\lambda_2 P_2 \end{bmatrix} \tag{8}$$

- **Hence, the system is asymptotically stable.**

Response of classical PI Controller in presence of time-varying disturbance



Motivation

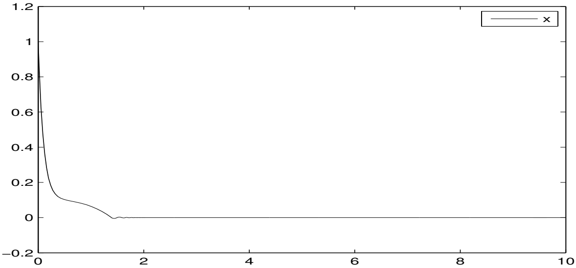
So, **What's next**??

- In order to reject the time varying but bounded disturbance, a non-smooth controller is introduced.
- The integral term of classical PI controller is replaced by an integral of discontinuous signum function.

$$u = -k_p x - k_I \int_0^{\tau} sign\left(x\left(\tau\right)\right) d\tau \qquad (9)$$

Thus the effective integral term of the controller is switching between two states.

Simulation of the proposed Controller in presence of time-varying disturbance



System and the Goal

Let us assume that we want to track a signal $x(t) = \alpha sin(t)$ in presence of the disturbance $1 + 3sin(t)$ where α is unknown but bounded.

Equations in terms of error $e(t) = x(t) - \alpha sin(t)$

$$\begin{aligned} \dot{e} &= \dot{x}(t) - \alpha cos(t) \\ &= u + 1 + 3sin(t) - \alpha cos(t) \\ &= -k_p e - k_I \int_0^{\tau} sign(e(\tau))\, d\tau + 1 + 3sin(t) - \alpha cos(t) \end{aligned} \tag{10}$$

Now our main goal is to show that error goes to zero with time which we will show using Lyapunov analysis.

It can also be noted that the system would be able to track any unknown signal as α is unknown and would reject any disturbance provided they are bounded.

Lyapunov Stability

Lyapunov stability theory is a standard tool and one of the most important tools in the analysis of nonlinear systems.

We do not need to solve the nonlinear differential equation to comment on the stability of the system.

Candidate for Lyapunov Function:

A function $V(x)$ is called a **Lyapunov Function** if:

- $V(x)$ and $\frac{\partial V(x)}{\partial x_i}$ are continuous in a region.
- $V(x)$ is positive definite in the region.
- relative to a system $\dot{x} = f(x)$, $\dot{V}(x)$ along a trajectory of the system is negative semi-definite in the region.

Philosophy of Lyapunov Direct Method and Stability

The basic philosophy of Lyapunov's direct method is the mathematical extension of a fundamental physical observation: if the total energy of a mechanical (or electrical) system is continuously dissipated, then the system, whether linear or nonlinear, must eventually settle down to an equilibrium point.

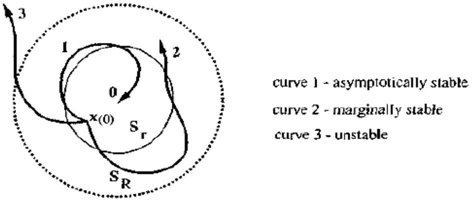


Figure: Concepts of stability

Notion of Stability in the sense of Lyapunov

Stability means that the system trajectory can be kept arbitrarily close to the origin by starting sufficiently close to it.

Formally, the definition states that the origin is stable, if, given that we do not want the state trajectory $x(t)$ to get out of a ball of arbitrarily specified radius B_R , a value $r(R)$ can be found such that starting the state from within the ball B_r at time 0 guarantees that the state will stay within the ball B_R thereafter.

An equilibrium point 0 is asymptotically stable if it is stable, and if in addition there exists some $r > 0$ such that $\|x(0)\| < r$ implies that $x(t) \rightarrow 0$ as $t \rightarrow \infty$

Change of Co-ordinates

Analysis using a trivial Lyapunov function remain inconclusive. Hence, we introduce a time varying change of variables to the system:

$$\dot{x}_1 = -k_1 x_1 + z, \ \dot{z} = -k_2 \text{sign}(x_1) + \dot{d} \tag{11}$$

By introducing time-varying change of variables

$$z_1(t) = \frac{x_1(t)}{L(t)}, \ z_2(t) = \frac{z(t)}{L(t)}, \ L(t) > 0, \quad \forall t \geq 0 \tag{12}$$

In the new Co-ordinates, system becomes:

$$\begin{aligned} \dot{z}_1 &= -\left(k_1 + \frac{\dot{L}}{L}\right) z_1 + z_2 \\ \dot{z}_2 &= -\frac{k_2}{L}\text{sign}(z_1) + \frac{\dot{d}}{L} - z_2\frac{\dot{L}}{L} \end{aligned} \tag{13}$$

Lyapunov Function Construction

We construct a non-trivial Lyapunov function by hit and trial method.

We come up with the following Lyapunov function

$$V(z_1, z_2) = \left(\pi_1 |z_1| + \frac{1}{2} z_2^2\right)^{\frac{3}{2}} + \pi_2 z_1 z_2 \tag{14}$$

We will show:

$$\begin{gathered} V(0,0) = 0 \\ V(z_1, z_2) > 0 \ \forall z_1, z_2 \neq 0 \\ \dot{V} \leq 0 \end{gathered} \tag{15}$$

Proof for 1^{st} and 2^{nd} condition of equation 15

Using Young's inequality we show that the proposed Lyapunov function is upper bounded above by zero.

$$
\begin{aligned}
V(z) &\geq (\pi_1 |z_1|)^{\frac{3}{2}} + \left(\frac{1}{2} z_2^2\right)^{\frac{3}{2}} \\
&- \pi_2 \left(\frac{2}{3} g^{\frac{3}{2}} |z_1|^{\frac{3}{2}} + \frac{1}{3} g^{-3} |z_2|^3\right), \quad g \geq 0 \\
&= \left(\pi_1^{\frac{3}{2}} - \frac{2}{3}\pi_2 g^{\frac{3}{2}}\right)|z_1|^{\frac{3}{2}} + \left(\left(\frac{1}{2}\right)^{\frac{3}{2}} - \frac{1}{3}\pi_2 g^{-3}\right)|z_2|^3 .
\end{aligned}
\tag{16}
$$

For $V \geq 0; \forall z$,

The condition would be $\pi_1 \geq \frac{2^{\frac{1}{2}} 2^{\frac{2}{3}}}{3} \pi_2$

Proof for 3^{rd} condition of equation 15.

Now our next aim is to show $\dot{V} < 0$,

$$\begin{aligned}\dot{V} &= \left\{ \frac{3}{2}\left(\pi_1 |z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}} \pi_1 \text{sign}(z_1) + \pi_2 z_2 \right\} \dot{z}_1 \\ &+ \left\{ \frac{3}{2}\left(\pi_1 |z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}} z_2 + \pi_2 z_1 \right\} \dot{z}_2 \\ &= -\frac{3}{2}\left(\pi_1 |z_1| + \frac{1}{2}z_2^2\right)\chi + \pi_2 z_2^2 - \pi_2\left(k_1 + \frac{\dot{L}}{L}\right) z_1 z_2 \\ &- \pi_2 \frac{k_2}{L}\text{sign}(z_1) z_1 + \pi_2 z_1 \frac{\dot{d}}{L} - \pi_2 z_1 z_2 \frac{\dot{L}}{L}\end{aligned}$$

where

$$\chi := \pi_1 \text{sign}(z_1)\left(\left(k_1 + \frac{\dot{L}}{L}\right) z_1 - z_2\right) + z_2\left(\frac{k_2}{L}\text{sign}(z_1) + z_2\frac{\dot{L}}{L} - \frac{\dot{d}}{L}\right).$$

Proof for 3^{rd} condition of equation 15. continued...

$$\dot{V} = -W_1(z)\left(\frac{\dot{L}}{L}\right) + W_2(z)\left(\frac{\dot{d}}{L}\right) - W_3^*(z) \tag{17}$$

where

$$\begin{aligned}
W_1(z) &= \frac{3}{2}\left(\pi_1|z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}\left(\pi_1|z_1| + z_2^2\right) + 2\pi_2 z_1 z_2 \\
W_2(z) &= \frac{3}{2}\left(\pi_1|z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}} z_2 + \pi_2 z_1 \\
W_3^*(z) &= \left(\frac{3}{2}\left(\pi_1|z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}} k_1\pi_1 + \pi_2\frac{k_2}{L}\right)|z_1| \\
&+ \frac{3}{2}\left(\pi_1|z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}\left(\frac{k_2}{L} - \pi_1\right)\operatorname{sign}(z_1 z_2)|z_2| \\
&- \pi_2 z_2^2 + \pi_2 k_1 z_1 z_2
\end{aligned} \tag{18}$$

Proof for 3^{rd} condition of equation 15. continued...

It can be shown that $W_3^*(z)$ would dominate over $W_2(z)$, given that $\left|\dot{d}\right| < k_2$.

$$
\begin{aligned}
W_1(z) &= \frac{3}{2}\left(\pi_1|z_1| + \frac{1}{2}z_2^2\right)^{\frac{1}{2}}\left(\pi_1|z_1| + z_2^2\right) + 2\pi_2 z_1 z_2 \\
&\geq \frac{3}{2}\left(\pi_1|z_1|\right)^{\frac{1}{2}}\pi_1|z_1| + \frac{3}{2}\left(\frac{1}{2}z_2^2\right)^{\frac{1}{2}} z_2^2 \\
&- 2\pi_2\left(\frac{2}{3}g^{\frac{3}{2}}|z_1|^{\frac{3}{2}} + \frac{1}{3}g^{-3}|z_2|^3\right) \\
&= \left(\frac{3}{2}\pi_1^{\frac{3}{2}} - \frac{4}{3}\pi_2 g^{\frac{3}{2}}\right)|z_1|^{\frac{3}{2}} + \left(\frac{3}{2^{\frac{3}{2}}} - \frac{2\pi_2}{3}g^{-3}\right)|z_2|^3
\end{aligned}
\tag{19}
$$

$W_1(z)$ is positive-definite if $\pi_1 > \frac{2^{\frac{5}{6}}2^2}{3^2}\pi_2$

Proof for 3^{rd} condition of equation 15. continued...

$W_3^*(z)$ dominates over $W_2(z)$, given that $\left|\dot{d}\right| < k_2$ and $W_1(z)$ is positive-definite

Hence the 3^{rd} condition of equation 15 is also proved.

Since, all the conditions of equation 15 are proved, We are in a position to state the Theorem.

Theorem

Consider the given system 11 and let $|\dot{d}| < k_2$, Then the system of differential equation 11 is asymptotically stable in spite of disturbance $\dot{d}$ if $k_1 > 0$ and $|\dot{d}| \leq k_2 \leq L(t)\left(\pi_1 + \frac{2^{\frac{3}{2}}}{3}\pi_2\right)$ with $\frac{2^2 2^{\frac{5}{6}}}{3^2}\pi_2 \leq \pi_1 \leq \frac{2^{\frac{3}{2}}}{3}\pi_2$ where $\pi_i > 0; i = 1,\ 2$ and $L(t), \dot{L}(t) > 0$.

Nonsmooth PI for the Higher Order Uncertain Chain of Integrators

For n^{th} order uncertain chain of integrators, the proposed nonsmooth PI controller is given as

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ &\vdots\\ \dot{x}_n &= f(t,x) + u + d,\end{aligned} \tag{20}$$

where $\mathbf{X}^\top = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix} \in \mathbb{R}^{1\times n}$ and u is the proposed control and is taken as

$$u = -f(t,x) - \mathbf{K_p X} - \int_0^t K_I \text{sign}\left(\mathbf{K_P X}\right) d\tau \tag{21}$$

Nonsmooth PI for the Higher Order Uncertain Chain of Integrators continued ...

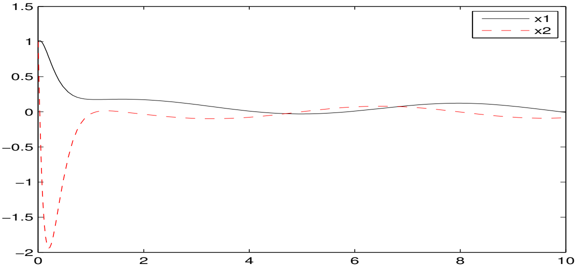
where $\mathbf{K_p} = \begin{bmatrix} k_1 & k_2 & \dots & k_n \end{bmatrix}$ with $k_i > 0$ for $i = 1, \dots, n$ and $K_I > 0$. After applying proposed controller equation 21 into the system 22, the closed loop system is given by

$$\begin{aligned} \dot{\mathbf{X}} &= \mathbf{AX} - \mathbf{BK_PX} + \mathbf{B}Z \\ \dot{Z} &= -K_I \text{sign}\left(\mathbf{K_PX}\right) + \dot{d} \end{aligned} \tag{22}$$

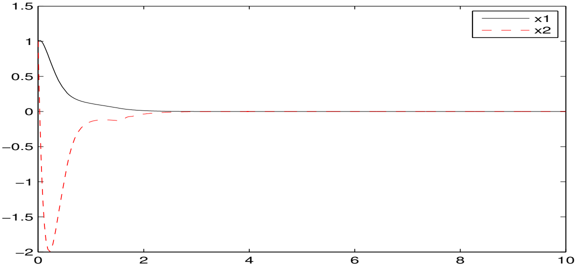
where, $Z = -\int_0^t K_I \text{sign}\left(\mathbf{K_PX}\right) d\tau + d$, and

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Response of the classical PI Controller in presence of time-varying disturbance for higher order system



Response of the proposed Controller in presence of time-varying disturbance for higher order system. continued. . .



Change of co-ordinates

On applying the following time-varying change of variables, $\mathbf{Z_1}(t) := \frac{\mathbf{X(t)}}{L(t)}$, $Z_2(t) := \frac{Z(t)}{L(t)}$, one can rewrite the (22) as

$$\begin{aligned} \dot{\mathbf{Z}}_\mathbf{1} &= -\left(\frac{\dot{L}}{L}\mathbf{I} + \mathbf{B}\mathbf{K}_\mathbf{P} - \mathbf{A}\right)\mathbf{Z_1} + \mathbf{B}Z_2 \\ \dot{Z}_2 &= -\frac{\dot{L}}{L}Z_2 - \frac{K_I}{L}\frac{\mathbf{K_P Z_1}}{\|\mathbf{K_P Z_1}\|} + \frac{\dot{d}}{L}, \end{aligned} \tag{23}$$

where $\mathbf{I}$ is an identity matrix and $L(t)$ is some continuously differentiable time varying positive function $\mathbb{C}^1$ i.e., $L\,(t) > 0 \quad \forall t \geq 0$ and $\dot{L} > 0$ exists.

Lyapunov Function Construction

Consider the $V(Z)$ be a Lyapunov function in the new co-ordinates

$$V(Z) = \left(\pi_1 ||\mathbf{Z_1}|| + \frac{1}{2} Z_2^2\right)^{\frac{3}{2}} + \pi_{\mathbf{2}} \mathbf{Z_1} Z_2, \tag{24}$$

where $Z := [\mathbf{Z_1}^\top \ Z_2]^\top$, $\pi_1 > 0$ and $\pi_{\mathbf{2}} = \begin{bmatrix} \pi_{21} & \pi_{22} & \dots & \pi_{2n} \end{bmatrix}$ with $\pi_{2i} > 0$ for $i = 1, \dots, n$. Equation 24 should satisfy all the conditions of 15 for the asymptotic Stability.

Proof for 1^{st} and 2^{nd} condition of equation 15

using Young's and norm inequalities, we are going to show that proposed Lyapunov function (7) is lower bounded from zero

$$
\begin{aligned}
V(Z) &\geq (\pi_1 \|\mathbf{Z_1}\|)^{\frac{3}{2}} + \left(\frac{1}{2} Z_2^2\right)^{\frac{3}{2}} \\
&- \|\pi_2\| \left(\frac{2}{3} g^{\frac{3}{2}} \|\mathbf{Z_1}\|^{\frac{3}{2}} + \frac{1}{3} g^{-3} |Z_2|^3\right), \quad g \geq 0 \\
&= \left(\pi_1^{\frac{3}{2}} - \frac{2}{3} \|\pi_2\| g^{\frac{3}{2}}\right) \|\mathbf{Z_1}\|^{\frac{3}{2}} + \left(\left(\frac{1}{2}\right)^{\frac{3}{2}} - \frac{1}{3} \|\pi_2\| g^{-3}\right) |Z_2|^3 .
\end{aligned}
\tag{25}
$$

For $V \geq 0; \forall z$,

The condition would be $\pi_1 \geq \frac{2^{\frac{1}{2}} 2^{\frac{2}{3}}}{3} \|\pi_2\|$

Proof for 3^{rd} condition of equation 15.

Now our next aim is to show $\dot{V} < 0$, Time derivative of Lyapunov function (24) along the system trajectory (23)

$$\begin{aligned}\dot{V}(Z) &= \left(\Theta\pi_1 \mathrm{SIGN}\left(\mathbf{Z_1}^\top\right) + \pi_2\mathbf{Z_2}\right)\dot{\mathbf{Z}}_\mathbf{1} + \left(\Theta Z_2 + \pi_2\mathbf{Z_1}\right)\dot{Z}_2 \\ &= \left(\Theta\pi_1 \mathrm{SIGN}\left(\mathbf{Z_1}^\top\right) + \pi_2\mathbf{Z_2}\right) \\ &\quad \left(-\left(\frac{\dot{L}}{L}\mathbf{I} + \mathbf{BK_P} - \mathbf{A}\right)\mathbf{Z_1} + \mathbf{B}Z_2\right) \\ &\quad + \left(\Theta Z_2 + \pi_2\mathbf{Z_1}\right)\left(-\frac{\dot{L}}{L}Z_2 - \frac{K_I}{L}\frac{\mathbf{K_P Z_1}}{\|\mathbf{K_P Z_1}\|} + \frac{\dot{d}}{L}\right),\end{aligned} \tag{26}$$

where $\Theta := \frac{3}{2}\left(\pi_1\|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}$

Proof for 3^{rd} condition of equation 15.continued . . .

$$\dot{V}(Z) = -W_1\left(\frac{\dot{L}}{L}\right) + W_2\left(\frac{\dot{d}}{L}\right) - W_3^*, \tag{27}$$

where,

$$W_1 = \frac{3}{2}\left(\pi_1 ||\mathbf{Z_1}|| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}\left(\pi_1 ||\mathbf{Z_1}|| + Z_2^2\right) + 2Z_2\pi_2\mathbf{Z_1} \tag{28a}$$

$$W_2 = \frac{3}{2}\left(\pi_1 ||\mathbf{Z_1}|| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}} Z_2 + \pi_2\mathbf{Z_1} \tag{28b}$$

$$W_3^* = \frac{3}{2}\left(\pi_1 \|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}} \Xi + \pi_2 Z_2 \left(\mathbf{BK_P} - \mathbf{A}\right)\mathbf{Z_1}$$
$$- Z_2^2\pi_2\mathbf{B} + \frac{K_I}{L}\pi_2\mathbf{Z_1}\text{sign}\left(\mathbf{K_P}\mathbf{Z_1}\right) \tag{28c}$$

where

$$\Xi := \pi_1\frac{\mathbf{Z_1}^\top\left(\mathbf{BK_P} - \mathbf{A}\right)\mathbf{Z_1}}{\|\mathbf{Z_1}\|} - \pi_1 Z_2\frac{\mathbf{Z_1}^\top\mathbf{B}}{\|\mathbf{Z_1}\|} + Z_2\frac{K_I}{L}\text{sign}\left(\mathbf{K_P}\mathbf{Z_1}\right).$$

Proof for 3^{rd} condition of equation 15.continued ...

It can be shown that $W_3^*(z)$ would dominate over $W_2(z)$, given that $\left|\dot{d}\right| < k_2$.

$$
\begin{aligned}
W_1(Z) &= \frac{3}{2}\left(\pi_1 \|\mathbf{Z_1}\| + \frac{1}{2}Z_2^2\right)^{\frac{1}{2}}\left(\pi_1 \|\mathbf{Z_1}\| + Z_2^2\right) + 2Z_2\pi_\mathbf{2}\mathbf{Z_1} \\
&\geq \frac{3}{2}\left(\pi_1 ||\mathbf{Z_1}||\right)^{\frac{1}{2}}\pi_1 ||\mathbf{Z_1}|| + \frac{3}{2}\left(\frac{1}{2}Z_2^2\right)^{\frac{1}{2}}Z_2^2 \\
&- 2||\pi_\mathbf{2}||\left(\frac{2}{3}g^{\frac{3}{2}}||\mathbf{Z_1}||^{\frac{3}{2}} + \frac{1}{3}g^{-3}|Z_2|^3\right) \\
&= \left(\frac{3}{2}\pi_1^{\frac{3}{2}} - \frac{4}{3}\|\pi_\mathbf{2}\| g^{\frac{3}{2}}\right)\|\mathbf{Z_1}\|^{\frac{3}{2}} + \left(\frac{3}{2^{\frac{3}{2}}} - \frac{2\|\pi_\mathbf{2}\|}{3}g^{-3}\right)|Z_2|^3
\end{aligned}
\tag{29}
$$

$W_1(z)$ is positive-definite if $\pi_1 > \frac{2^{\frac{5}{6}}2^2}{3^2}\|\pi_\mathbf{2}\|$

Proof for 3^{rd} condition of equation 15. continued...

$W_3^*(z)$ dominates over $W_2(z)$, given that $\left|\dot{d}\right| < k_I$ and $W_1(z)$ is positive-definite

Hence the 3^{rd} condition of equation 15 is also proved.

Since, all the conditions of equation 15 are proved, We are in a position to state the Theorem.

Theorem

Consider the system 20 and let $|\dot{d}| < K_I$. Then the system of differential inclusion 20 is asymptotically stable in spite of disturbance d if $\mathbf{K_P}$ is selected such that $Q := (\mathbf{BK_P} - \mathbf{A})$ has positive eigenvalues and

$\left|\dot{d}\right| \leq K_I \leq L(t) \left(-\pi_1 \|\mathbf{B}\| + \frac{2^{\frac{3}{2}}}{3} \pi_\mathbf{2} \mathbf{B}\right)$ with

$\frac{2^2 2^{\frac{5}{6}}}{3^2} \|\pi_\mathbf{2}\| \leq \pi_1 \leq \frac{2^{\frac{3}{2}}}{3} \|\pi_\mathbf{2}\|$ where $\pi_1 > 0$, $L(t), \dot{L}(t) > 0$ and $\pi_\mathbf{2} = \begin{bmatrix} \pi_{21} & \pi_{22} & \dots & \pi_{2n} \end{bmatrix}$ with $\pi_{2i} > 0$ for $i = 1, \dots, n$.

Outputs and Future Goals

FutureGoals :
1. Effectiveness of the proposed controller when the disturbance is **stochastic** in nature.
2. Consideration of Actuator dynamics.

Output

- Submitted a manuscript entitled 'Strict Lyapunov Function for system with Nonsmooth PI Controller' in the 'IEEE Transactions on Automatic Control'.

Thank you for your attention